

An inequality for Jacobi polynomials of form $P_n^{(\alpha_n, \beta_n)}(x)$

Zhulin He*

Department of Statistics, Iowa State University, Ames, IA 50011, USA

Yuyuan Ouyang

Department of Mathematical Sciences, Clemson University, Clemson, SC 29634, USA

Abstract

We prove an inequality for Jacobi polynomials that

$$\Delta_n(x) := P_n^{(\alpha_n, \beta_n)}(x)P_n^{(\alpha_{n+1}, \beta_{n+1})}(x) - P_{n-1}^{(\alpha_n, \beta_n)}(x)P_{n+1}^{(\alpha_{n+1}, \beta_{n+1})}(x) \leq 0, \quad \forall x \geq 1,$$

where $\alpha_n = an$ and $\beta_n = bn$ for some $a, b \geq 0$. The above inequality has a similar taste as the Turán type inequalities, but with α_n and β_n that depends linearly on n .

Keywords: Turán inequality, Jacobi polynomials

1. Introduction

The Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ are a class of orthogonal polynomials that are well studied in many literatures. The polynomial representation with real variable x is

$$P_n^{(\alpha, \beta)}(x) = \sum_{t=0}^n \binom{n+\alpha}{n-t} \binom{n+\beta}{t} \left(\frac{x-1}{2}\right)^t \left(\frac{x+1}{2}\right)^{n-t}. \quad (1)$$

In [1], it was proved that

$$\Delta_n(x) := R_n^2(x) - R_{n-1}(x)R_{n+1}(x) \geq \frac{(\beta - \alpha)(1 - x)}{2(n + \alpha + 1)(n + \beta)} R_n^2(x), \quad \forall x \in [-1, 1], n \geq 1, \alpha, \beta > -1, \quad (2)$$

where

$$R_n(x) := \frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(1)}.$$

*Corresponding author

Email address: hezhulin@iastate.edu (Zhulin He)

Consequently, we have $\Delta_n(x) \geq 0$ for all $x \in [-1, 1]$, $n \geq 1$, $\beta \geq \alpha > -1$. Such result is known as a Turán type inequality that originates from the studies of Legendre polynomials by Turán in [2] (see also [3]). It should be noted that the discussion is restricted to $x \in [-1, 1]$ here because that the Jacobi polynomials are orthogonal in $[-1, 1]$. The definition we use in (1) is in fact well defined for any $x \in \mathbb{R}$.

In this study, we will prove the following inequality for Jacobi polynomials:

$$\Delta_n(x) := P_n^{(\alpha_n, \beta_n)}(x)P_n^{(\alpha_{n+1}, \beta_{n+1})}(x) - P_{n-1}^{(\alpha_n, \beta_n)}(x)P_{n+1}^{(\alpha_{n+1}, \beta_{n+1})}(x) \leq 0, \quad \forall x \in [-1, 1]. \quad (3)$$

Here, α_n and β_n are dependent on n with

$$\alpha_n = an, \beta_n = bn \quad (4)$$

for some $a, b \geq 0$. The inequality (3) is different from (2) due to such dependence on n . It should also be noted that unlike (2) we are not considering $x \in [-1, 1]$ in (3). This is because that polynomials $\{P_n^{(\alpha_n, \beta_n)}(x)\}_{n=0}^{\infty}$ are in general not orthogonal on $[-1, 1]$.

2. Notations and preliminaries

For easy reading, we will use the following notations:

$$P_l^k := P_l^{(\alpha_k, \beta_k)}(x), (P_l^k)' := \frac{d}{dx} P_l^{(\alpha_k, \beta_k)}(x).$$

Under the above notation, we have

$$\Delta_n(x) = P_n^n P_n^{n+1} - P_{n-1}^n P_{n+1}^{n+1}. \quad (5)$$

We will also use notations Δ_n and Δ'_n without x for convenience.

The recurrence formula for differentiation (see, e.g., Section 4.5 in [4]) is an important tool for our analysis. In particular, we have

$$\begin{aligned} (1-x^2)(P_n^n)' &= A_n P_n^n + B_n P_{n-1}^n \\ (1-x^2)(P_n^{n+1})' &= C_n P_n^{n+1} + D_n P_{n+1}^{n+1} \end{aligned}$$

where

$$\begin{aligned} A_n &= -n \left(x + \frac{\beta_n - \alpha_n}{2n + \alpha_n + \beta_n} \right), \\ B_n &= \frac{2(n + \alpha_n)(n + \beta_n)}{2n + \alpha_n + \beta_n}, \\ C_n &= (n + \alpha_{n+1} + \beta_{n+1} + 1) \left(x - \frac{\beta_{n+1} - \alpha_{n+1}}{2n + \alpha_{n+1} + \beta_{n+1} + 2} \right), \\ D_n &= -\frac{2(n+1)(n + \alpha_{n+1} + \beta_{n+1} + 1)}{2n + \alpha_{n+1} + \beta_{n+1} + 2}. \end{aligned}$$

Substituting the definition of α_n and β_n in (4) to the above equations, we have

$$(1 - x^2) (P_n^n)' = nAP_n^n + nBP_{n-1}^n \quad (6)$$

$$(1 - x^2) (P_n^{n+1})' = (n+1)CP_n^{n+1} + (n+1)DP_{n+1}^{n+1} \quad (7)$$

where

$$A = -x - \frac{b-a}{2+a+b}, B = \frac{2(1+a)(1+b)}{2+a+b}, C = (1+a+b) \left(x - \frac{b-a}{2+a+b} \right), \text{ and } D = -\frac{2(1+a+b)}{2+a+b}. \quad (8)$$

Note that (6) and (7) are polynomial equalities. Therefore, although most studies of Jacobi polynomials focus on the case when $x \in [-1, 1]$, the relations (6) and (7) hold for all $x \in \mathbb{R}$.

Since Jacobi polynomials $P_n^{\alpha, \beta}(x)$ with fixed α and β are orthogonal polynomials, they satisfy the following important inequality (see, e.g., (3.3.6) in [4]):

$$(P_{n+1}^{\alpha, \beta}(x))' P_n^{\alpha, \beta}(x) - (P_n^{\alpha, \beta}(x))' P_{n+1}^{\alpha, \beta}(x) > 0, \quad \forall x \in \mathbb{R}.$$

Setting $\alpha = \alpha_{n+1}$ and $\beta = \beta_{n+1}$ in the above equation, we have

$$P_n^{n+1} (P_{n+1}^{n+1})' - P_{n+1}^{n+1} (P_n^{n+1})' > 0. \quad (9)$$

The above inequality will be useful in our proof of (3).

15 3. Main results

We will prove (3) in this section. The general idea of our proof is to show that there exists r and s such that at all critical points of the function $f_n^{r,s}(x) := (x-1)^r(x+1)^s \Delta_n(x)$ in $(1, \infty)$, the

values of $f_n^{r,s}(x)$ are all non-positive. As a consequence, we have $f_n^{r,s}(x) \leq 0$ in $[1, \infty)$, and hence $\Delta_n \leq 0$. Similar idea was used in the proofs of orthogonal polynomial inequality relations in [5, 6]. Noting that

$$\frac{d}{dx} f_n^{r,s}(x) = (x-1)^{r-1} (x+1)^{s-1} \{ [r(x+1) + s(x-1)] \Delta_n + (x^2-1) \Delta'_n \}, \quad (10)$$

any critical points $x > 1$ can be characterized by the following relationship between Δ_n and Δ'_n :

$$[r(x+1) + s(x-1)] \Delta_n + (x^2-1) \Delta'_n = 0.$$

To start with, we prove a technical lemma below that is related to the above equation.

Lemma 1.

$$E_n \Delta_n + (x^2-1) \Delta'_n = (x^2-1) \left[\frac{P_n^n (P_{n+1}^{n+1})'}{n+1} - \frac{(P_n^n)' P_{n+1}^{n+1}}{n} \right], \quad (11)$$

where

$$E_n := (n+1)A + nC. \quad (12)$$

Proof. By (5), (6) and (7) we have

$$\begin{aligned} & (1-x^2) \left[\Delta'_n - \frac{P_n^n (P_{n+1}^{n+1})'}{n+1} + \frac{(P_n^n)' P_{n+1}^{n+1}}{n} \right] \\ &= (1-x^2) \left[\frac{n+1}{n} (P_n^n)' P_{n+1}^{n+1} + \frac{n}{n+1} P_n^n (P_{n+1}^{n+1})' - (P_{n-1}^n)' P_{n+1}^{n+1} - P_{n-1}^n (P_{n+1}^{n+1})' \right] \\ &= \frac{n+1}{n} (nA P_n^n + nB P_{n-1}^n) P_{n+1}^{n+1} + \frac{n}{n+1} ((n+1)C P_n^{n+1} + (n+1)D P_{n+1}^{n+1}) P_n^n \\ &\quad - (nC P_{n-1}^n + nD P_n^n) P_{n+1}^{n+1} - ((n+1)A P_{n+1}^{n+1} + (n+1)B P_n^{n+1}) P_{n-1}^n \\ &= ((n+1)A + nC) P_n^n P_{n+1}^{n+1} - ((n+1)A + nC) P_{n-1}^n P_{n+1}^{n+1}. \end{aligned}$$

We conclude (11) immediately from the above result and (12). \square

We make one observation from the above lemma. Applying (8) to (12), we have

$$E_n = (an + bn - 1)x + \frac{[(2+a+b)n+1](b-a)}{2+a+b}.$$

Therefore, we can obtain

$$r(x+1) + s(x-1) = E_n \quad (13)$$

by setting

$$\begin{aligned} r &= \frac{1}{2} \left[(an + bn - 1)x + \frac{[(2 + a + b)n + 1](b - a)}{2 + a + b} \right], \\ s &= \frac{1}{2} \left[(an + bn - 1)x - \frac{[(2 + a + b)n + 1](b - a)}{2 + a + b} \right]. \end{aligned} \quad (14)$$

We are now ready to prove (3).

Theorem 1. *For all $x \geq 1$, we always have*

$$\Delta_n(x) \leq 0.$$

Here the inequality becomes equality if and only if $x = 1$.

Proof. By (6) and (7) we have

$$\begin{aligned} & (1 - x^2) \left(\frac{P_n^n (P_{n+1}^{n+1})'}{n + 1} - \frac{P_{n+1}^{n+1} (P_n^n)'}{n} \right) \\ &= P_n^n (AP_{n+1}^{n+1} + BP_n^{n+1}) - P_{n+1}^{n+1} (AP_n^n + BP_{n-1}^n) \\ &= B(P_n^n P_n^{n+1} - P_{n-1}^n P_{n+1}^{n+1}). \end{aligned}$$

Using the above relation and noting the definition of Δ_n in (5), we have

$$\Delta_n = \frac{1 - x^2}{B} \left(\frac{P_n^n (P_{n+1}^{n+1})'}{n + 1} - \frac{P_{n+1}^{n+1} (P_n^n)'}{n} \right). \quad (15)$$

Let us define

$$f_n^{r,s}(x) := (x - 1)^r (x + 1)^s \Delta_n(x), \quad (16)$$

where r and s are defined in (14), so that (13) holds. Clearly, the sign of $\Delta_n(x)$ and $f_n^{r,s}(x)$ are the same in $[1, \infty)$. Noting the derivative of $f_n^{r,s}$ in (10), the relations (11), and (13), we have

$$\begin{aligned} \frac{d}{dx} f_n^{r,s}(x) &= (x - 1)^{r-1} (x + 1)^{s-1} (E_n \Delta_n - (1 - x^2) \Delta_n') \\ &= (x - 1)^r (x + 1)^s \left[\frac{P_n^n (P_{n+1}^{n+1})'}{n + 1} - \frac{(P_n^n)' P_{n+1}^{n+1}}{n} \right]. \end{aligned}$$

Therefore, for any critical points of $f_n^{r,s}(x)$ in $(1, \infty)$, we always have

$$\frac{P_n^n (P_{n+1}^{n+1})'}{n + 1} = \frac{(P_n^n)' P_{n+1}^{n+1}}{n}.$$

Suppose that $x > 1$ is a critical point $f_n^{a,b}(x)$ in $(1, \infty)$. Noting the definition of $f_n^{a,b}$ in (16), using the above relation, (9), and (15), we obtain

$$\begin{aligned} f_n^{r,s}(x) &= -\frac{(x-1)^{r+1}(x+1)^{s+1}}{B} \left(\frac{P_n^n (P_{n+1}^{n+1})'}{n+1} - \frac{P_{n+1}^{n+1} (P_n^n)'}{n} \right) \\ &= -\frac{(x-1)^{r+1}(x+1)^{s+1}}{BP_n^{n+1}} \left(\frac{P_n^n (P_{n+1}^{n+1})' P_n^{n+1}}{n+1} - \frac{P_{n+1}^{n+1} (P_n^n)' P_n^{n+1}}{n} \right) \\ &= -\frac{(x-1)^{r+1}(x+1)^{s+1} P_n^n}{B(n+1)P_n^{n+1}} \left(P_n^{n+1} (P_{n+1}^{n+1})' - P_{n+1}^{n+1} (P_n^n)' \right). \end{aligned}$$

Since $x > 1$, by (1) we have in the above relation that $P_{n+1}^{n+1}, P_n^{n+1} > 0$. Therefore, applying (9) to the above, we obtain $f_n^{r,s}(x) < 0$ at any critical points in $(1, \infty)$. If $f_n^{r,s}(1) \leq 0$, then we conclude that $f_n^{r,s}(x) \leq 0$ for all $x > 1$, and so $\Delta_n < 0$. To finish the proof, it suffices to check the value of $f_n^{r,s}$ when $x = 1$ and $x \rightarrow \infty$. In fact, by (1) we have

$$\begin{aligned} \Delta_n(1) &= P_n^{\alpha_n, \beta_n}(1) P_n^{\alpha_{n+1}, \beta_{n+1}}(1) - P_{n-1}^{\alpha_n, \beta_n}(1) P_{n+1}^{\alpha_{n+1}, \beta_{n+1}}(1) \\ &= \binom{n+\alpha_n}{n} \binom{n+\alpha_{n+1}}{n} - \binom{n-1+\alpha_n}{n-1} \binom{n+1+\alpha_{n+1}}{n+1} \\ &= \frac{n+\alpha_n}{n} \binom{n+\alpha_n-1}{n-1} \binom{n+\alpha_{n+1}}{n} - \frac{n+1+\alpha_{n+1}}{n+1} \binom{n-1+\alpha_n}{n-1} \binom{n+\alpha_{n+1}}{n} \\ &= 0. \end{aligned}$$

Here the last equality is from the definition of α_n in (4). When $x \rightarrow \infty$, note that the coefficient of the leading term x^{2n} in $\Delta_n(x)$ is

$$\begin{aligned} &\frac{(2n+\alpha_n+\beta_n+1)!}{2^n n! (n+\alpha_n+\beta_n+1)!} \frac{(2n+\alpha_{n+1}+\beta_{n+1}+1)!}{2^n n! (n+\alpha_{n+1}+\beta_{n+1}+1)!} \\ &- \frac{(2n+\alpha_n+\beta_n-1)!}{2^{n-1} (n-1)! (n+\alpha_n+\beta_n)!} \frac{(2n+\alpha_{n+1}+\beta_{n+1}+3)!}{2^{n+1} (n+1)! (n+\alpha_{n+1}+\beta_{n+1}+2)!} \\ &= \frac{(2n+\alpha_n+\beta_n-1)! (2n+\alpha_{n+1}+\beta_{n+1}+1)!}{2^{2n} (n-1)! n! (n+\alpha_n+\beta_n)! (n+\alpha_{n+1}+\beta_{n+1}+1)!} \\ &\quad \left[\frac{(2n+\alpha_n+\beta_n)(2n+\alpha_n+\beta_n+1)}{n(n+\alpha_n+\beta_n+1)} - \frac{(2n+\alpha_{n+1}+\beta_{n+1}+2)(2n+\alpha_{n+1}+\beta_{n+1}+3)}{(n+1)(n+\alpha_{n+1}+\beta_{n+1}+2)} \right], \end{aligned}$$

in which substituting (4) we have

$$\begin{aligned}
& \frac{(2n + \alpha_n + \beta_n)(2n + \alpha_n + \beta_n + 1)}{n(n + \alpha_n + \beta_n + 1)} - \frac{(2n + \alpha_{n+1} + \beta_{n+1} + 2)(2n + \alpha_{n+1} + \beta_{n+1} + 3)}{(n + 1)(n + \alpha_{n+1} + \beta_{n+1} + 2)} \\
&= \frac{(2 + a + b)((2 + a + b)n + 1)}{(1 + a + b)n + 1} - \frac{(2 + a + b)((2 + a + b)(n + 1) + 1)}{(1 + a + b)(n + 1) + 1} \\
&= (2 + a + b) \left[\frac{1}{1 + a + b + \frac{1}{n}} - \frac{1}{1 + a + b + \frac{1}{n + 1}} \right] < 0.
\end{aligned}$$

25 Thus we have $\lim_{x \rightarrow \infty} \Delta_n(x) = -\infty$. Therefore, we conclude that $f_n^{a,b}(x) < 0$ for all $x > 1$, and hence $\Delta_n(x) < 0$ as well. \square

4. Concluding remarks

In this note we prove a proof of Turán-like inequality for Jacobi polynomials of form $P_n^{\alpha_n, \beta_n}(x)$. The difference between our result (3) and previous ones in the literature is that in our case α_n and β_n are not constants but depends linearly on n . While our result applies for all variables $x \geq 1$, it will be interesting to see if similar inequality holds for x in $[-1, 1]$ and $(-\infty, -1]$.

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